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GENERALIZED COMPLEX PERRON NUMBERS

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Abstract

The classical notion of a Perron number is extended to generalized complex Perron number. Some algebraic and topological properties of this class of algebraic numbers are described.

1. Introduction and Basic Properties of Generalized Complex Perron Numbers

Some intensely studied classes of real positive algebraic integers are described by the sizes of their conjugates, e.g., Pisot, Salem, Garsia or Perron numbers. In the past decades, some of these concepts were extended in various directions, for instance, to algebraic numbers (extended Salem numbers (Dubickas and Smyth [4])), to complex numbers (complex Pisot numbers (Smyth [15]) and complex Garsia numbers (Hare and Panju [6])), and to function fields (Pisot and Salem numbers (Scheicher [13])).

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Here, we are concerned with an extension of the notion of a Perron number to complex numbers. We recall that the name 'Perron number' was coined by Lind [9] in his studies of entropies of topological Markov shifts. Perron numbers constitute the spectral radii of certain integral matrices (see, e.g., [11, Subsection 11.1]). Roughly speaking, we can say that the arithmetic properties of Perron numbers resemble those of nonnegative rational integers. It is worth mentioning that the celebrated Perron-Frobenius theorem states that a square matrix with positive real entries has a unique largest real eigenvalue and that the corresponding eigenvector has strictly positive components; moreover, a similar statement for certain classes of nonnegative matrices is established. These facts have far reaching applications, for instance, to the theory of dynamical systems and to probability theory.

Let us first recall that a Perron number is a real positive algebraic integer, which is strictly larger in modulus than its other conjugates (see, e.g., [11, Definition 11.1.2]). The following extension to complex numbers was introduced by Thurston [16].

Definition 1. A complex Perron number is a nonzero algebraic integer α, which is strictly larger in modulus than the other roots of its minimal polynomial, except for $\overline{\alpha}$, its complex conjugate.

Thurston [16] proved that the expansion of a self-similar tiling is a complex Perron number. Further, he mentioned without proof that for each complex Perron number, there exists a tiling. Indeed, Kenyon [7] showed that for each complex Perron number α, there is a self-similar tiling with expansion α. For an introduction to this topic and historical remarks, the reader is referred to [1, 7].

Let us denote by $\mathcal{P}$, the set of complex Perron numbers.

Example 2. (i) Every nonzero integer is a complex Perron number, we have

$$(-\mathcal{P}) \cup \mathcal{P}^\star \cup (-\mathcal{P}^\star) \subset \mathcal{P}, \quad \mathcal{P} \cap \mathbb{R} \subset \mathcal{P}^\star \cup (-\mathcal{P}^\star), \quad \text{and} \quad \mathcal{P} \cap \mathbb{R}_{>0} = \mathcal{P}^\star.$$

Here, we denote by $\mathcal{P}^\star$ the set of Perron numbers.

(ii) Let $k \in \mathbb{N}, k \geq 2,$ and $\mathrm{i} := \sqrt{-1}.$ Then $\pm\sqrt{k}\mathrm{i} \in \mathcal{P},$ but $k^{1/d} \notin \mathcal{P}$ for all $d \geq 2.$

(iii) Let ζ be a root of unity, $d \geq 3, k \in \mathbb{Z},$ and $|k|^{1/d} \notin \mathbb{N}.$ Then $\zeta|k|^{1/d} \notin \mathcal{P}.$

(iv) ([12, Lemma 3.3]) Let $\alpha \in \mathcal{P}$ with $|\alpha| = \sqrt{2}.$ Then

$$\alpha \in \left\{\pm 1 \pm \mathrm{i}, \pm\sqrt{2}\mathrm{i}, \pm\frac{1}{2} \pm \frac{\sqrt{7}}{2}\mathrm{i}\right\}.$$

(v) Boyd [2, Supplement] calculated

$$\min\left\{\overline{|\alpha|} : \alpha \text{ algebraic integer of degree } d,\ \alpha \text{ not a root of unity}\right\},$$

for small d; as customary the symbol $\overline{|\alpha|}$, the house of α, denotes the maximal modulus of the conjugates of α. Based on these results, Table 1 below gives the minimal polynomials P_d of complex Perron numbers of smallest house $\mu(d)$ of degree $d \leq 5$. We denote by ν_d the number of roots outside the unit disc of the polynomial P_d and by r_d the number of its real roots. Finally, θ_0 is the smallest Pisot number, i.e., the real root of the cubic polynomial $X^3 - X - 1$.

Table 1. Minimal polynomials of smallest complex Perron numbers of degree at most 5

d	P_d	$\mu(d)$	ν_d	r_d
1	$X - 1$	1	0	1
2	$X^2 + 1$	$\lvert\mathrm{i}\rvert = 1$	0	0
3	$X^3 + X^2 - 1$	$\theta_0^{1/2} = 1.15096\ldots$	2	1
4	$X^4 + X^3 + 1$	$1.18375\ldots$	2	0
5	$X^5 - X^3 - X^2 + X + 1$	$1.12164\ldots$	4	1

We now collect some results on complex Perron numbers.

Lemma 3 ([12, Lemma 2.5])**.** *Let* $\alpha \in \mathbb{C}$ *satisfy* $\alpha^n \in \mathcal{P}$ *for all sufficiently large* n. *Then* $\alpha \in \mathcal{P}$.

The following lemma slightly extends Example 2 (iv):

Lemma 4 ([12, Lemma 3.1])**.** *Let* $k, d \in \mathbb{N}_{>0}$ *and* $\alpha \in \mathbb{C}$ *such that* $|\alpha| = k^{1/d}$.

(i) *If* $d = 2$, *then* $\alpha \in \mathcal{P}$ *if and only if* α *is an integer*, *or a non-real quadratic integer*.

(ii) *If* $d \neq 2$, *then* $\alpha \in \mathcal{P}$ *if and only if* α *is an integer*.

Proof. This is [12, Lemma 3.1] in case $d = 2$, and the other cases are trivial. □

According to a proposal by Dubickas and Smyth [4] (see also [5]), we introduce the following extension of the notion of a complex Perron number:

Definition 5. The nonzero algebraic number α is called a *generalized complex Perron number*, if it is strictly larger in modulus than the other roots of its minimal polynomial, except for $\overline{\alpha}$, its complex conjugate.

A criterion for the product of two complex Perron numbers to be a complex Perron number can be given under rather restrictive conditions.

Theorem 6. *Let* α *and* β *be generalized complex Perron numbers such that the degree* $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}]$ *is prime*. *Then* $\alpha\beta$ *is a generalized complex Perron number*.

Proof. Without loss of generality, we may assume that α and β are algebraic integers. If $\alpha \in \mathbb{Z}$ or $\beta \in \mathbb{Z}$, then the assertion is clear be Example 2. Thus, we may assume $\alpha, \beta \notin \mathbb{Z}$, hence $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\alpha)$. Let $\sigma_1, \dots, \sigma_p$ be the embeddings of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$, where $\sigma_1(\alpha) = \alpha$. As β is a linear combination of the powers of α with rational coefficients, we also have $\sigma_1(\beta) = \beta$; furthermore, $\sigma_i(\alpha)$ is the complex conjugate of α, if and only if $\sigma_i(\beta)$ is the complex conjugate of β. Now, we easily deduce our assertion. □

Remark 7. Theorem 6 does not hold for fields of non-prime degree (for instance, take $\alpha = \mathrm{i}$ and $\beta = \sqrt{2}\mathrm{i}$ (see Example 2)).

Among other things, Lind [9] studied arithmetic properties of the set of Perron numbers. In general, these properties cannot be carried over to complex Perron numbers as our next remark shows.

Remark 8. (i) Unlike real Perron numbers, the set $\mathcal{P}$ is neither additively nor multiplicatively closed: For instance, setting $\gamma = \sqrt{2}\mathrm{i}$, we have $\sqrt{2} + \gamma \in \mathcal{P}$, but $(\sqrt{2} + \gamma) - \gamma \notin \mathcal{P}$, and $\sqrt{3}\mathrm{i} \in \mathcal{P}$, but $\gamma\sqrt{3}\mathrm{i} = -\sqrt{6} \notin \mathcal{P}$ (cf. [9, Proposition 1]). Further, $\alpha^n \in \mathcal{P}$ does not necessarily imply $\alpha \in \mathcal{P}$: For instance, take $\alpha = \sqrt{2}$ and $n = 2$.

(ii) In contrast to real Perron numbers, in general, the equality $\mathbb{Q}(\alpha + \beta) = \mathbb{Q}(\alpha, \beta)$ does not hold for complex Perron numbers α, β: For instance, take $\alpha = -\beta \in \mathcal{P} \setminus \mathbb{Z}$.

(iii) If α, β and $\alpha\beta$ are complex Perron numbers, then in general, neither α nor β belong to $\mathbb{Q}(\alpha\beta)$ as the example $\alpha = \beta = \sqrt{2}\mathrm{i}$ shows.

(iv) Kenyon [7] exploited the non-real complex Perron numbers, which satisfy a quadrinomial of the form

$$X^d - aX^{d-1} + bX + c, \quad (a, b, c \in \mathbb{N}, c \geq 1, d \geq 3),$$

for his construction of self-similar tilings.

2. Generalized Weak Complex Perron Numbers

Weak Perron numbers were introduced by Lind [10, p. 590] in the course of his studies of spectral radii of matrices with nonnegative integer entries. A description of weak Perron numbers can be found in [3]. In this section, we extend this notion to complex algebraic numbers.

Definition 9. The algebraic number α is called

• a weak complex Perron number, if α is an algebraic integer and α^n is a complex Perron number for some positive integer n;

• a generalized weak complex Perron number, if α^n is a generalized complex Perron number for some positive integer n.

Now, we carry over some algebraic and topological properties of Perron numbers to the set $\mathcal{P}_w$ of weak complex Perron numbers (cf. [9, Section 5]). We showed in [3] that the algebraic integer is a weak Perron number, if and only if it coincides with its house. The analogue of this property yields a characterization of weak complex Perron numbers.

Theorem 10. *The algebraic number* α *is a generalized weak complex Perron number*, *if and only if* $|\alpha| = \overline{|\alpha|}$.

Proof. Let α be a generalized weak complex Perron number, hence

$$|\alpha|^n = |\alpha^n| = \overline{|\alpha^n|},$$

for some $n \in \mathbb{N}_{>0}$, and therefore $|\alpha| = \overline{|\alpha|}$. To prove the converse, we proceed by induction on

$$\mu(\alpha) := \operatorname{Card}\left\{i \in \{1, \dots, d\} : |\alpha_i| = \overline{|\alpha|}\right\},$$

where d is the degree and $\alpha_i = \alpha_1\ \alpha_2, \dots, \alpha_d$ are the conjugates of α. If $\mu(\alpha) = 1$, then α is a generalized complex Perron number and we are done. Let $m := \mu(\alpha) > 1$ and order the conjugates of α in the following way:

$$|\alpha_1| = \cdots = |\alpha_m| > |\alpha_{m+1}| \geq \cdots \geq |\alpha_d|.$$

Then $\beta := \alpha / \alpha_m$ is an algebraic number with the properties $|\beta| = \overline{|\beta|}$ and $\mu(\beta) < \mu(\alpha)$, thus by induction hypothesis, β^n is a generalized complex

Perron number for some $n \in \mathbb{N}_{>0}$. Let $(\alpha^n)_j$ be a conjugate of α^n, which is different from α^n and its complex conjugate. Then, we have

$$|\alpha^n| = |(\beta\alpha_m)^n| = |\beta^n||\alpha_m^n| > \left|\frac{(\alpha^n)_j}{\alpha_m^n}\right||\alpha_m^n| = |(\alpha^n)_j|,$$

thus α^n is a generalized complex Perron number. The proof is completed. □

The theorem allows the following conclusions:

Corollary 11. *Every algebraic number* (*integer*, *respectively*) *has a conjugate*, *which is a generalized complex Perron number* (*generalized weak complex Perron number, respectively*).

Corollary 12. *Every real positive algebraic integer is a weak complex Perron number*, *if and only if it is a K-number in the sense of Korec* [8].

Proof. Clear by [14, Theorem]. □

Corollary 13. *For every* $d \in \mathbb{N}_{>0}$, *the set*

$$\{|\alpha| : \alpha \in \mathcal{P}_w,\ \deg(\alpha) \leq d\}$$

is a discrete subset of $[1, \infty)$.

Proof. Similarly as the proof of [9, Proposition 3]. □

Our final statements are straightforward extensions of well-known facts. In particular, Theorem 14 (iii) goes back to a result of Schinzel [14].

Theorem 14. (i) *The following statements are equivalent for an algebraic number* α:

(a) α *is a generalized weak complex Perron number*.

(b) *There exists a positive rational integer q such that* $mq\alpha$ *is a weak complex Perron number for all* $m \in \mathbb{N}_{>0}$.

(c) *There exists a positive rational integer* q *such that* $q\alpha$ *is a weak complex Perron number*.

(ii) *If* α *be a generalized complex weak Perron number*, *then every* $n \in \mathbb{N}_{>0}$ *some conjugate of* $\alpha^{1/n}$ *is a generalized weak Perron number*.

(iii) *If* α, β *are generalized weak Perron numbers with* $\alpha\beta \in \mathbb{Q}$, *then there exists some* $n \in \mathbb{N}_{>0}$ *such that* $\alpha^n \in \mathbb{Q}$ *and* $\beta^n \in \mathbb{Q}$.

Proof. (i) (a) $\Rightarrow$ (b) Let $n \in \mathbb{N}_{>0}$ such that α^n is strictly larger than all its other algebraic conjugates apart from its complex conjugate. Pick $q \in \mathbb{N}_{>0}$ such that $q\alpha$ is an algebraic integer. For each $m \in \mathbb{N}_{>0}$, the modulus of the algebraic integer $(mq\alpha)^n$ dominates all its other conjugates, thus, we have $mq\alpha \in \mathcal{P}_w$ by Theorem 10.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) Trivial.

(ii) This is clear by (i) and Theorem 10.

(iii) We closely the proof of [14, Corollary 3]. Let $\sigma : \mathbb{Q}(\alpha, \beta) \to \mathbb{C}$ be a $\mathbb{Q}$-monomorphism, hence $|\alpha| \geq |\sigma(\alpha)|$ and $|\beta| \geq |\sigma(\beta)|$ by Theorem 10. With $q := \alpha\beta \in \mathbb{Q}$, we find

$$|q| \geq |\sigma(\alpha)|\,|\sigma(\beta)| = |\sigma(\alpha\beta)| = |q|,$$

hence $|\sigma(\alpha)| = |\alpha|$ and $|\sigma(\beta)| = |\beta|$. Therefore, $a := |\alpha|^d \in \mathbb{Q}_{>0}$, where d is the degree of α. We conclude that the conjugates of α can be written in the form $\zeta^j a^{1/d}$ with some primitive d-th root of unity ζ and some $j \in \{1, \ldots, d\}$. This implies $\alpha^d \in \mathbb{Q}$. Similarly, we deduce $\beta^e \in \mathbb{Q}$, where e is the degree of β. Finally, we observe $\alpha^{de}, \beta^{de} \in \mathbb{Q}$. $\square$

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